

OHSTPY-HEP-T-96-032

DOE/ER/01545-699

HUEAP-010

Mesons in 2-Dimensional QCD on the Light Cone

Osamu Abe^{*†}, Gordon J. Aubrecht II[‡], and K. Tanaka[§]

Department of Physics

The Ohio State University

174 West 18th Avenue

Columbus, Ohio 43210

Abstract

Two dimensional QCD is quantized on the light front coordinate. We solve the Einstein-Schrödinger equation by the use of Tamm-Dancoff truncation and find that the simplest wavefunction produces the M/g versus m/g relation in agreement with other calculations, where M and m are the masses of the ground state and quarks, respectively.

11.10.Kk, 11.15.Pg, 11.15.Tk

* Permanent address: Physics Laboratory, Asahikawa Campus, Hokkaido University of Education, 9 Hokumoncho, Asahikawa 070, Japan

† e-mail address: osamu@asa.hokkyodai.ac.jp

‡ e-mail address: aubrecht@mps.ohio-state.edu

§ e-mail address: tanaka@mps.ohio-state.edu

I. INTRODUCTION

The light front Tamm-Dancoff (LFTD) method [1] has been introduced as an alternative tool to lattice gauge theory to investigate relativistic bound states nonperturbatively. The LFTD is a Tamm-Dancoff (T-D) approximation [2] applied to field theories quantized on the light front coordinate. In the usual perturbative field theory quantized in the equal time frame, a vacuum state is an infinite sea of constituents such as electrons and photons in QED and quarks and gluons in QCD. Bound states of a hydrogen atom in QED, and mesons and baryons in QCD, arise as excitations of this sea.

The T-D method independently considered the possibility of describing the vacuum and the bound states with a finite number of particles, and solving a set of coupled integral equations. This method was applied to a variety of problems in strong interactions, but it was unsuccessful because a large number of amplitudes was required to solve a given problem.

On the other hand, in the LFTD method all constituents have non-negative longitudinal momenta defined by $p^+ = \frac{p^0 + p^3}{\sqrt{2}}$ in the light front coordinate. Then the vacuum of the system under consideration can not have constituents, or the physical vacuum is equivalent to the bare vacuum in the light front coordinate. Therefore, we may expect this approach to remove a serious problem in the T-D approximation in the equal time frame.

It was found in several problems such as the hydrogen atom, positronium, and the two-dimensional Yukawa interaction, that one needed an additional T-D amplitude to obtain the particle spectrum [3]. We have provided an argument [4] for why this happens to be the case.

In 1974, 't Hooft [5] introduced a model, 2 dimensional QCD, that continues to be studied. We examine this model in the framework of the LFTD method.

II. MESONS IN 2D QCD

We consider the quantized version of Einstein's equation

$$2P(H_0 + H_I)|\Psi\rangle = M^2|\Psi\rangle, \quad (2.1)$$

in two dimensional QCD with $SU(N_c)$ symmetry, for states with mass M on the light front coordinate. Here P is the momentum operator in the $SU(N_c)$ gauge, H_0 is the free part of the Hamiltonian, given by $H_0 \equiv P_{free}^- + P_{self}^-$, and $H_I = P_0^- + P_2^-$ is the interaction Hamiltonian.

We consider a meson state $|\Psi(f, \bar{f}'; p)\rangle$ with momentum p , which is described by

$$\begin{aligned} |\Psi(f, \bar{f}'; p)\rangle = & \sum_c \int_0^\infty \frac{dq_1}{\sqrt{2\pi q_1}} \frac{dq_2}{\sqrt{2\pi q_2}} \delta(p - q_1 - q_2) \psi_2(f, \bar{f}'; q_1, q_2) b^\dagger(f, q_1, c) d^\dagger(f', q_2, c) |0\rangle \\ & + \sum_{f_1, f_2, f_3, f_4} \sum_{c_1, c_2, c_3, c_4} \int_0^\infty \frac{dq_1}{\sqrt{2\pi q_1}} \frac{dq_2}{\sqrt{2\pi q_2}} \frac{dq_3}{\sqrt{2\pi q_3}} \frac{dq_4}{\sqrt{2\pi q_4}} \delta(p - q_1 - q_2 - q_3 - q_4) \\ & \times \psi_4(f, \bar{f}'; f_1, f_2, \bar{f}_3, \bar{f}_4; q_1, q_2, q_3, q_4; c_1, c_2, c_3, c_4) \\ & \times b^\dagger(f_1, q_1, c_1) b^\dagger(f_2, q_2, c_2) d^\dagger(f_3, q_3, c_3) d^\dagger(f_4, q_4, c_4) |0\rangle. \end{aligned} \quad (2.2)$$

The first order LFTD approximation results from putting Eq. (2.2) into Eq. (2.1) and projecting the resultant equation onto a state with a fermion and an anti-fermion, and a state with two fermions and two anti-fermions. This leads to

$$\langle f_1, q_1, c_1; \bar{f}_2, q_2, c_2 | (M^2 - 2PH_0) |\Psi\rangle = \langle f_1, q_1, c_1; \bar{f}_2, q_2, c_2 | 2PH_I |\Psi\rangle, \quad (2.3)$$

and

$$\begin{aligned} & \langle f_1, q_1, c_1; f_2, q_2, c_2; \bar{f}_3, q_3, c_3; \bar{f}_4, q_4, c_4 | (M^2 - 2PH_0) |\Psi\rangle \\ & = \langle f_1, q_1, c_1; f_2, q_2, c_2; \bar{f}_3, q_3, c_3; \bar{f}_4, q_4, c_4 | 2PH_I |\Psi\rangle, \end{aligned} \quad (2.4)$$

where

$$|f_1, q_1, c_1; \bar{f}_2, q_2, c_2\rangle = \frac{b^\dagger(f_1, q_1, c_1)}{\sqrt{2\pi q_1}} \frac{d^\dagger(f_2, q_2, c_2)}{\sqrt{2\pi q_2}} |0\rangle, \quad (2.5)$$

and

$$\begin{aligned}
& |f_1, q_1, c_1; f_2, q_2, c_2; \bar{f}_3, q_3, c_3; \bar{f}_4, q_4, c_4 \rangle \\
&= \frac{b^\dagger(f_1, q_1, c_1)}{\sqrt{2\pi q_1}} \frac{b^\dagger(f_2, q_2, c_2)}{\sqrt{2\pi q_2}} \frac{d^\dagger(f_3, q_3, c_3)}{\sqrt{2\pi q_3}} \frac{d^\dagger(f_4, q_4, c_4)}{\sqrt{2\pi q_4}} |0 \rangle.
\end{aligned} \tag{2.6}$$

It is straightforward to derive coupled integral equations for the amplitudes ψ_2 and ψ_4 .

The results are

$$\begin{aligned}
& \delta_{f_1 f} \delta_{f_2 f'} \delta_{c_1 c_2} F(M; f, f'; q_1, q_2) \psi_2(f, \bar{f}'; q_1, q_2) \\
&= -\delta_{f_1 f} \delta_{f_2 f'} \delta_{c_1 c_2} \frac{N_c^2 - 1}{N_c} \frac{pg^2}{2\pi} \wp \int_{-\infty}^{\infty} \frac{dk}{k^2} \psi_2(f, \bar{f}'; q_1 + k, q_2 - k) \\
&- \frac{2pg^2}{2\pi} \sum_{j: \text{flavors}} \sum_{d_1, d_2, d_3: \text{colors}} \mathcal{T}_{c_1 d_2; d_3 d_1} \wp \int_{-\infty}^{\infty} \frac{dk}{k^2} \int_0^{\infty} dq \\
&\times \psi_4(f, \bar{f}'; [j, f_1], [\bar{j}, \bar{f}_2]; [q, q_1 + k], [-q - k, q_2]; [d_1, d_2], [d_3, c_2]) \\
&- \frac{2pg^2}{2\pi} \sum_{j: \text{flavors}} \sum_{d_1, d_2, d_3: \text{colors}} \mathcal{T}_{d_3 d_1; d_2 c_2} \wp \int_{-\infty}^{\infty} \frac{dk}{k^2} \int_0^{\infty} dq \\
&\times \psi_4(f, \bar{f}'; [j, f_1], [\bar{j}, \bar{f}_2]; [q, q_1], [q_2 - k, -q + k]; [d_1, c_1], [d_2, d_3]),
\end{aligned} \tag{2.7}$$

where \wp stands for principal value integral, N_c is the order of the gauge group, and we have used

$$\mathcal{T}_{cd;ef} = \frac{1}{2} \left\{ \delta_{cf} \delta_{de} - \frac{1}{N_c} \delta_{cd} \delta_{ef} \right\}, \tag{2.8}$$

$$\begin{aligned}
& F(M; f_1, f_2 \cdots; q_1, q_2 \cdots) \\
&= M^2 - 2(q_1 + q_2 + \cdots) \left\{ \frac{m_{f_1}^2 + \alpha(q_1)}{2q_1} + \frac{m_{f_2}^2 + \alpha(q_2)}{2q_2} + \cdots \right\},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
& \psi_4(f, \bar{f}'; [f_1, f_2], [\bar{f}_3, \bar{f}_4]; [q_1, q_2], [q_3, q_4]; [c_1, c_2], [c_3, c_4]) \\
&= \psi_4(f, \bar{f}'; f_1, f_2, \bar{f}_3, \bar{f}_4; q_1, q_2, q_3, q_4; c_1, c_2, c_3, c_4) - (1 \leftrightarrow 2) - (3 \leftrightarrow 4) \\
&+ (1 \leftrightarrow 2, 3 \leftrightarrow 4),
\end{aligned} \tag{2.10}$$

and

$$F(M; f_1, f_2, f_3, f_4; q_1, q_2, q_3, q_4) \psi_4(f, \bar{f}'; [f_1, f_2], [\bar{f}_3, \bar{f}_4]; [q_1, q_2], [q_3, q_4]; [c_1, c_2], [c_3, c_4])$$

$$\begin{aligned}
&= \frac{2pg^2}{2\pi} \left[\left\{ \delta_{f_1 f} \delta_{f_2 f_3} \delta_{f_4 f'} \mathcal{T}_{c_1 c_4; c_2 c_3} \psi_2(f, \bar{f}'; q_1 + q_2 + q_3, q_4) \cdot \frac{1}{(q_2 + q_3)^2} \right. \right. \\
&- (1 \leftrightarrow 2) - (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \Big\} \\
&- \left\{ \delta_{f_1 f_4} \delta_{f_2 f} \delta_{f_3 f'} \mathcal{T}_{c_2 c_3; c_1 c_4} \psi_2(f, \bar{f}'; q_2, q_1 + q_3 + q_4) \cdot \frac{1}{(q_1 + q_4)^2} \right. \\
&- (1 \leftrightarrow 2) - (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \Big\} \Big]. \tag{2.11}
\end{aligned}$$

The α term in Eq. (2.9) denotes the meson self energy and is found to be equal to

$$-(N_c^2 - 1)g^2/(2N_c\pi).$$

Hereafter, we restrict ourselves to the one flavor case. We define

$$p = q_1 + q_2, \quad q_1 = \left(\frac{1}{2} + x\right)p, \quad \psi_2(q_1, q_2) \equiv b_0(x), \quad a = \frac{1}{2} + x \quad \text{and} \quad b = \frac{1}{2} - x. \tag{2.12}$$

Substituting Eq. (2.11) into Eq. (2.7), we have

$$\left[M^2 - \frac{4(m^2 + \alpha)}{1 - 4x^2} \right] b_0(x) = I_0(x) + I_1(x) + I_2(x), \tag{2.13}$$

where

$$I_0(x) \equiv -2\mathcal{N}_{1\wp} \int_{-a}^b \frac{dy}{y^2} b_0(x + y), \tag{2.14}$$

$$\begin{aligned}
I_1(x) &\equiv \mathcal{N}_{2\wp} \int_{-a}^0 \frac{dy}{y^2} \int_0^{-y} dz \frac{1}{F(M; z, a + y, -z - y, b)} \times \\
&\quad \left[\left\{ \frac{n}{(a - z)^2} + \frac{1}{y^2} \right\} b_0(x) + \left\{ \frac{n}{(b + z)^2} + \frac{1}{(1 + y)^2} \right\} b_0(y + z + \frac{1}{2}) \right. \\
&\quad \left. - \left\{ \frac{n}{(b + z)^2} + \frac{1}{y^2} \right\} b_0(x + y) - \left\{ \frac{n}{(a - z)^2} + \frac{1}{(1 + y)^2} \right\} b_0(z - \frac{1}{2}) \right], \tag{2.15}
\end{aligned}$$

and

$$\begin{aligned}
I_2(x) &\equiv -\mathcal{N}_{2\wp} \int_0^b \frac{dy}{y^2} \int_0^y dz \frac{1}{F(M; -z + y, a, b - y, z)} \\
&\quad \left[\left\{ \frac{n}{(b - z)^2} + \frac{1}{(1 - y)^2} \right\} b_0(\frac{1}{2} - z) + \left\{ \frac{n}{(a + z)^2} + \frac{1}{y^2} \right\} b_0(x + y) \right. \\
&\quad \left. - \left\{ \frac{n}{(b - z)^2} + \frac{1}{y^2} \right\} b_0(x) - \left\{ \frac{n}{(a + z)^2} + \frac{1}{(1 - y)^2} \right\} b_0(-\frac{1}{2} + y - z) \right]. \tag{2.16}
\end{aligned}$$

Here we use the shorthand notation

$$\mathcal{N}_1 = (N_c^2 - 1)g^2/(4N_c\pi), \quad \mathcal{N}_2 = (N_c - 1)g^4/(4N_c\pi^2), \quad \text{and} \quad n = (N_c + 1)/N_c. \quad (2.17)$$

With the definition of Eqs. (2.15) and (2.16) we can show

$$I_2(-x) = \pm I_1(x), \quad (2.18)$$

corresponding to the symmetry property of $b_0(\pm x) = \pm b_0(x)$. It is not possible to solve Eq. (2.13) analytically. One usually selects a wavefunction $b_0(x)$ and tries to match both sides of Eq. (2.13). It happens that the results do not depend sensitively on the form of $b_0(x)$. In other words, different forms of $b_0(x)$ lead to similar predictions. Further, the contributions of the integrals $I_1(x)$ and $I_2(x)$ are at most 10% of that of $I_0(x)$. Suppose we take the simple form of power series expansion for b_0 . Since we wish to determine the ground state wavefunction, which has no node (and thus no odd powers of x), we write

$$b_0(x) = 1 + \bar{A}x^2 + \bar{B}x^4 + \bar{C}x^6 + \dots \quad (2.19)$$

The form of Eq. (2.19) is motivated by our assumption that $b_0(x)$ is an even function and we normalize the first term to 1. The power series is expressible in terms of a linear combination of Airy functions that are relevant to linear potentials. It is surprising, as we shall show, that most of the information is present in the first terms $1 + \bar{A}x^2$ of (2.19) and that higher order terms such as the \bar{B} and \bar{C} terms provide a small correction. We expect \bar{A} to be negative in order that $b_0(x)$ decrease away from $x = 0$. The wavefunction $1 - |\bar{A}|x^2$ is a maximum at $x = 0$ and is zero at $x = \pm|\bar{A}|^{-1/2}$.

As a warmup, consider the simplest model [6] of (2.13) with only $I_0(x)$ and

$$b_0(x) = 1 + \bar{A}x^2. \quad (2.20)$$

We study this model because it contains the ingredients of the resulting relation of the ground state mass M and the quark mass m . The additional contributions of $I_1(x)$, $I_2(x)$ and higher order terms such as \bar{B} and \bar{C} terms provide a small correction to this relation.

Define

$$\tilde{M}^2 = M^2/2N_1 \quad \tilde{m}^2 = \left(\frac{m^2}{2N_1}\right) - 1, \quad (2.21)$$

where the relation $\alpha = -2N_1$ is used, and write the simplified (2.13) as

$$\left\{(1 - 4x^2)\tilde{M}^2 - 4\tilde{m}^2\right\}(1 + \bar{A}x^2) = 4 + \bar{A}\left\{4x^2 - 2(1 - 4x^2)x \ln \frac{1 - 2x}{1 + 2x} - 1 + 4x^2\right\}. \quad (2.22)$$

From equating the coefficients of the x^0 and x^2 terms, we obtain

$$\tilde{M}^2 - 4\tilde{m}^2 = 4 - \bar{A}, \quad (2.23)$$

$$4\tilde{M}^2 - (\tilde{M}^2 - 4\tilde{m}^2)\bar{A} = -16\bar{A} \quad (2.24)$$

We find that when $b_0(x) = 1$, Eqs. (2.23) and (2.24) yield $M = m = 0$, a relation we regard as a boundary condition. When we eliminate \bar{A} from (2.23) and (2.24), we obtain

$$M/g = \sqrt{6} m/g = 2.45 m/g \quad (2.25)$$

and the condition M^2 be real is $m/g \leq 2/\sqrt{3\pi}$. Also, one can express \tilde{M}^2 and \tilde{m}^2 in terms of \bar{A} ,

$$\frac{M^2}{m^2} = \frac{3 + \bar{A}/4}{1/2 + \bar{A}/16}. \quad (2.26)$$

We return to $I_0(x)$ and $I_1(x)$ given by Eqs. (2.14) and (2.15), respectively. The $I_0(x)$ is written in the form

$$\begin{aligned} I_0(x) &= \frac{8\mathcal{N}_1}{1 - 4x^2} + 2\bar{A}\mathcal{N}_1 \left\{ \frac{4x^2}{1 - 4x^2} - 2x \log \frac{1 - 2x}{1 + 2x} - 1 \right\} \\ &+ 2\bar{B}\mathcal{N}_1 \left\{ \frac{4x^4}{1 - 4x^2} - 4x^3 \log \frac{1 - 2x}{1 + 2x} - \frac{1}{12} - 3x^2 \right\} \\ &+ 2\bar{C}\mathcal{N}_1 \left\{ \frac{4x^6}{1 - 4x^2} - 6x^5 \log \frac{1 - 2x}{1 + 2x} - \frac{1}{80} - \frac{x^2}{4} - 5x^4 \right\} + \dots, \end{aligned} \quad (2.27)$$

and

$$I_1(x) = \bar{A}I_A(x) + \bar{B}I_B(x) + \bar{C}I_C(x) + \dots. \quad (2.28)$$

Here

$$I_A(x) = \mathcal{N}_2 \int_{-a}^0 dy \frac{E(x, y)}{y^2} \int_0^{-y} dz \times \left\{ 1 + \frac{C(x, y)}{z(z+y) - C(x, y)} \right\} \left[2n \left\{ \frac{x}{a-z} + \frac{x+y}{b+z} \right\} - \frac{2x}{y} + \frac{2z-1}{1+y} \right], \quad (2.29)$$

$$I_B(x) = \mathcal{N}_{2\wp} \int_{-a}^0 dy \frac{E(x, y)}{y^2} \int_0^{-y} dz \left[1 + \frac{C(x, y)}{z(z+y) - C(x, y)} \right] \times \left[n \left\{ -\frac{z-b}{z-a} \left(x^2 + \left(z - \frac{1}{2} \right)^2 \right) + \frac{2y+z+a}{z+b} \left(\left(y+z + \frac{1}{2} \right)^2 + (x+y)^2 \right) \right\} - \frac{1}{y} (2x+y) \left(x^2 + (x+y)^2 \right) + \frac{1}{1+y} (y+2z) \left(\left(y+z + \frac{1}{2} \right)^2 + \left(z - \frac{1}{2} \right)^2 \right) \right], \quad (2.30)$$

and

$$I_C(x) = \mathcal{N}_{2\wp} \int_{-a}^0 dy \frac{E(x, y)}{y^2} \int_0^{-y} dz \left[1 + \frac{C(x, y)}{z(z+y) - C(x, y)} \right] \times \left[n \left\{ -\frac{z-b}{z-a} \left(x^4 + x^2 \left(z - \frac{1}{2} \right)^2 + \left(z - \frac{1}{2} \right)^4 \right) + \frac{2y+z+a}{z+b} \left(\left(y+z + \frac{1}{2} \right)^4 + \left(y+z + \frac{1}{2} \right)^2 (x+y)^2 + (x+y)^4 \right) \right\} - \frac{1}{y} (2x+y) \left(x^4 + x^2 (x+y)^2 + (x+y)^4 \right) + \frac{1}{1+y} (y+2z) \left(\left(y+z + \frac{1}{2} \right)^4 + \left(y+z + \frac{1}{2} \right)^2 \left(z - \frac{1}{2} \right)^2 + \left(z - \frac{1}{2} \right)^4 \right) \right] \quad (2.31)$$

where the functions E and C are defined as

$$E(x, y) = \frac{b(a+y)}{M^2 b(a+y) - (m^2 + \alpha)(1+y)}, \quad (2.32)$$

and

$$C(x, y) = \frac{(m^2 + \alpha)(a+y)by}{M^2 b(a+y) - (m^2 + \alpha)(1+y)}. \quad (2.33)$$

We may fit the integrands of $I_A(x)$, $I_B(x)$, and $I_C(x)$ empirically after performing the z -integral:

$$\text{Integrand of } I_{A,B,C}(x) = f_{1A,B,C}(y;r) \left(x + \frac{y}{2}\right) + f_{3A,B,C}(y;r) \left(x + \frac{y}{2}\right)^3, \quad (2.34)$$

We return to $I_0(x)$ and $I_1(x)$ given by Eqs. (2.14) and (2.15), respectively, that may be written in the form where $r \equiv M^2/(m^2 + \alpha)$ and details of the functions f_1 and f_3 are given in the Appendix. This procedure produces equations that may be solved sequentially at each order of x for the unknown parameters \bar{A} , \bar{B} , \bar{C} , \dots , r , and $\tilde{g} = \frac{g^2}{m^2 + \alpha}$,

$$\begin{aligned} \mathcal{O}(x^0) : \quad r - 4 &= -\frac{4\tilde{g}}{3\pi} \left[-4 + \bar{A} + \frac{\bar{B}}{12} + \frac{\bar{C}}{80} \right] \\ &+ \frac{\tilde{g}^2}{6\pi^2} \left[\bar{A}F_{A0} + \bar{B}F_{B0} + \bar{C}F_{C0} \right], \end{aligned} \quad (2.35)$$

$$\begin{aligned} \mathcal{O}(x^2) : \quad (r - 4)\bar{A} - 4r &= -\frac{4\tilde{g}}{3\pi} \left[-16\bar{A} + \frac{8\bar{B}}{3} + \frac{\bar{C}}{5} \right] \\ &+ \frac{\tilde{g}^2}{6\pi^2} \left[\bar{A}(F_{A2} - 4F_{A0}) + \bar{B}(F_{B2} - 4F_{B0}) + \bar{C}(F_{C2} - 4F_{C0}) \right], \end{aligned} \quad (2.36)$$

$$\begin{aligned} \mathcal{O}(x^4) : \quad (r - 4)\bar{B} - 4r\bar{A} &= -\frac{4\tilde{g}}{3\pi} \left[\frac{64\bar{A}}{3} - 32\bar{B} + 4\bar{C} \right] \\ &+ \frac{\tilde{g}^2}{6\pi^2} \left[\bar{A}(F_{A4} - 4F_{A2}) + \bar{B}(F_{B4} - 4F_{B2}) + \bar{C}(F_{C4} - 4F_{C2}) \right], \end{aligned} \quad (2.37)$$

$$\begin{aligned} \mathcal{O}(x^6) : \quad (r - 4)\bar{C} - 4r\bar{B} &= -\frac{4\tilde{g}}{3\pi} \left[\frac{256\bar{A}}{15} + \frac{128\bar{B}}{3} - 48\bar{C} \right] \\ &+ \frac{\tilde{g}^2}{6\pi^2} \left[\bar{A}(F_{A6} - 4F_{A4}) + \bar{B}(F_{B6} - 4F_{B4}) + \bar{C}(F_{C6} - 4F_{C4}) \right]. \end{aligned} \quad (2.38)$$

To solve to order x^2 , we neglect the terms in the expansion of the wave function to higher order that involve \bar{B} , \bar{C} , etc. This yields two equations from (2.35) and (2.36) in terms of \bar{A} , r , and \tilde{g} . The solution to order x^4 involves neglecting terms \bar{C} , etc., in the wavefunction and using (2.35)-(2.37), and so on.

We can solve Eqs. (2.35)-(2.38) numerically for r in terms of \tilde{g} . We show the result in Fig.1. For small values of m/g , we shall show that in the present model M/g approximately satisfies Eq. (2.25).

III. VALIDITY OF THE APPROXIMATION.

In this section we consider the contributions of $I_1(x)$ and $I_2(x)$ or alternatively $I_A(x)$ (2.29), $I_B(x)$ (2.30), and $I_C(x)$ (2.31) to the ground state mass M .

At first, we neglect $I_1(x)$ and $I_2(x)$ in Eq. (2.13). We can solve the above equations (2.35)-(2.38) numerically for r in terms of \tilde{g} when we neglect these contributions of the integrals $I_A(x)$, $I_B(x)$, and $I_C(x)$, \dots . Multiplying Eq. (2.13) by $1 - 4x^2$ and setting $\bar{D} = \dots = 0$ for simplicity, we have

$$\mathcal{O}(x^0) \quad : \quad r - 4 = -\frac{4\tilde{g}}{3\pi} \left[-4 + \bar{A} + \frac{\bar{B}}{12} + \frac{\bar{C}}{80} \right], \quad (3.1)$$

$$\mathcal{O}(x^2) \quad : \quad (r - 4)\bar{A} - 4r = -\frac{4\tilde{g}}{3\pi} \left[-16\bar{A} + \frac{8\bar{B}}{3} + \frac{\bar{C}}{5} \right], \quad (3.2)$$

$$\mathcal{O}(x^4) \quad : \quad (r - 4)\bar{B} - 4r\bar{A} = -\frac{4\tilde{g}}{3\pi} \left[\frac{64\bar{A}}{3} - 32\bar{B} + 4\bar{C} \right], \quad (3.3)$$

$$\mathcal{O}(x^6) \quad : \quad (r - 4)\bar{C} - 4r\bar{B} = -\frac{4\tilde{g}}{3\pi} \left[\frac{256\bar{A}}{15} + \frac{128\bar{B}}{3} - 48\bar{C} \right]. \quad (3.4)$$

Then (3.1) to (3.2) with $C = 0$, after eliminating r and \tilde{g} , yield

$$12\bar{A}^3 + 144\bar{A}^2 + 256\bar{A} - 44\bar{A}\bar{B} + \bar{A}^2\bar{B} - 336\bar{B} - \bar{B}^2 = 0. \quad (3.5)$$

Solving the above equation for \bar{A} , we obtain three roots $\bar{A}_1(\bar{B})$,

$\bar{A}_2(\bar{B})$, and $\bar{A}_3(\bar{B})$. The solutions where \bar{A} and \bar{B} are real is shown in Fig. 2. The requirement that M is real rules out the solution \bar{A}_3 in Fig. 2.

We can express M^2 and m^2 in terms of \bar{A} , \bar{B} with the aid of (3.1), (3.2) and (3.3):

$$\frac{M^2}{m^2} = \left(3 + \frac{\bar{A}}{4} - \frac{2}{3} \frac{\bar{B}}{\bar{A}} + \frac{\bar{B}}{48} \right) / \left(\frac{1}{2} + \frac{\bar{A}}{16} - \frac{3\bar{B}}{16\bar{A}} + \frac{\bar{B}}{192} \right). \quad (3.6)$$

We may also repeat this calculation, by including (3.4), to find \bar{C} .

Our range of m/g is limited to $0 - 2/\sqrt{3\pi}$. We have included these results in Fig. 1. For small values of m/g , M/g approximately satisfies (2.25) (dashed, lowest line).

We may repeat the procedures given above, but including the integrals $I_A(x)$, $I_B(x)$, and $I_C(x)$. The changes from the solution of Sec. 3 and those calculated immediately above are under 10% for all the cases. Overall, Fig. 1 shows that M/g vs. m/g differs at most by less than 20% from the values obtained from the simplest model.

IV. COMPARISON TO OTHER PARAMETRIZATIONS.

We compare the relation M/m when different wavefunctions are used for $b_0(x)$. We begin by writing 't Hooft's equation for a two particle state [5] in the present notation,

$$\frac{\pi M^2}{g^2} b_0(x) = \left(\frac{1}{1/2 - x} + \frac{1}{1/2 + x} \right) \left(\frac{\pi m^2}{g^2} - 1 \right) b_0(x) - \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \frac{b_0(y)}{(y - x)^2}. \quad (4.1)$$

Eq. (4.1) corresponds to our Eq. (2.13). Both sides of Eq. (4.1) are integrated with respect to x , and we obtain

$$M^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} b_0(x) dx = 4m^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \frac{b_0(x)}{(1 - 4x^2)}. \quad (4.2)$$

When we substitute 't Hooft's wavefunction $\sin(n\pi(x + 1/2)) = \cos\left(n\pi x + \frac{(n-1)\pi}{2}\right)$ in (2.28), we find by inspection that only the odd numbers of n contribute to Eq. (4.2) and obtain

$$2M^2/n\pi = 2m^2 \text{Si}(n\pi), \quad n = 1, 3, 5, \dots, \quad (4.3)$$

where the sine integral $\text{Si}(z)$ is

$$\text{Si}(z) = \int_0^z dt \frac{\sin t}{t}. \quad (4.4)$$

For $n = 1$

$$M^2 = \pi \text{Si}(\pi) m^2 = 5.82 m^2, \quad \text{or} \quad M/g = 2.41 m/g. \quad (4.5)$$

which is in good agreement with (2.25).

If a wavefunctions of the form [7] $b_0(x) = (1 - 4x^2)^\beta$ is used, where $\beta > 0$ and for the range of small β , we obtain $b_0(x) \approx 1 - 4\beta x^2$. Upon comparison with (2.20) we get $\bar{A} = -4\beta$. One has for such wavefunctions $(1 - 4x^2)^\beta$ the boundary condition [5, 7]

$$\frac{2\pi m^2}{g^2 \left(N_c - \frac{1}{N_c}\right)} - 1 + \pi\beta \cot(\pi\beta) = 0. \quad (4.6)$$

In the limit $N_c \rightarrow \infty$, we get $\tan \pi\beta = \pi\beta$, which holds when β is small.

For small β and $N_c = 3$, Eq. (4.6) is given by

$$3\pi m^2/4g^2 - \pi^2\beta^2(1 + 3\pi m^2/8g^2)/3 = 0, \quad (4.7)$$

so that

$$\bar{A} = -4\beta = -6m(1 + 3\pi m^2/8g^2)^{-1/2}/\sqrt{\pi}g. \quad (4.8)$$

We substitute Eq. (4.8) in Eq. (2.26) and obtain

$$\frac{M^2}{m^2} = 6 \left[\frac{(1 + 3\pi m^2/8g^2)^{1/2} - m/2\sqrt{\pi}g}{(1 + 3\pi m^2/8g^2)^{1/2} - 3m/4\sqrt{\pi}g} \right], \quad (4.9)$$

with the aid of $0 \leq m/g \leq 2/\sqrt{3\pi}$. We calculate Eq. (4.9) and find

$$2.45 \frac{m}{g} \leq \frac{M}{g} \leq 2.56 \frac{m}{g}. \quad (4.10)$$

The average slope of the numerical result by Hornbostel et al. [8], after a correction, is

$$M/g = 3.1 m/g. \quad (4.11)$$

The SU(2) lattice gauge result of Hamer[9] is

$$M/g = 3.2 m/g. \quad (4.12)$$

A comparison of Eqs. (2.25), (4.5), (4.10), (4.11) and (4.12), indicates that the ratios differ by about 30%.

Fig. 3 shows the comparison of the results of these models. It is apparent that all the models give values that are in general agreement. Thus, our simplest model $b_0(x) = 1 + \bar{A}x^2$ is enough to obtain the approximate lowest ground state mass.

V. SUMMARY AND CONCLUSIONS

We have studied the ratio of the ground state mass M to the quark mass m . The wavefunction $b_0(x)$ of (2.13) is expanded in a power series of even powers of x . We then matched equal powers of x on both side of the equation (2.13). We have examined how each term of (2.19) affects the final result. When $b_0(x)$ is set equal to 1, we find to order x^2 that $M = m = 0$.

Next, when $b_0(x) = 1 + Ax^2$, the x^0 and x^2 equations yield the result (2.25), $M/g = \sqrt{6}m/g$. The substitution of $b_0(x) = \sin n\pi(x + 1/2)$ in 't Hooft's equation gives for the ground state ($n=1$), $M/g = [\pi \text{Si}(\pi)]^{1/2}m/g = 2.41 m/g$, where $\text{Si}(x)$ is the sine integral.

The simplest model of the wavefunction seems adequate to describe the features of the two dimensional light front description of mesons. The additional terms of the wavefunction $b_0(x)$ provide small corrections.

ACKNOWLEDGEMENT

One of us (O.A.) would like to thank Department of Physics, the Ohio State University for their hospitality, where this work was completed.

APPENDIX. DESCRIPTION OF THE FIT FOR $I_A(x)$, $I_B(x)$ AND $I_C(x)$

In general, our functions $f_{1A,B,C,\dots}$ and $f_{3A,B,C,\dots}$ are parametrized in the form

$$f_{ij}(y; r) = \sum_k^n b_{ij,k}(r) \frac{(-y)^{k/2}}{(\gamma_{ij}y + \delta_{ij})^{p_{ij}}(-y)^{q_{ij}}}, \quad (\text{A.1})$$

with different values of the parameters as shown in Table 1. The r -dependence of $b_{ij,k}(r)$ is quadratic.

Table 1: Values of parameters

	n	γ	δ	p	q
f_{1A}	3	0	1	1	0
f_{1B}	8	1	0	0	0
f_{1C}	10	1	0	0	5
f_{3A}	8	1	1	2	0
f_{3B}	3	0	1	1	0
f_{3C}	6	1	0	0	5

REFERENCES

- [1] K. G. Wilson, in *Lattice '89*, Proceedings of the International Symposium, Capri, Italy, 1989, edited by R. Petronzio *et al.* [Nucl. Phys. **B** (Proc. Suppl.) **17** (1990)].
- [2] I. Tamm, J. Phys.(USSR) **9**, 449 (1945),
S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
- [3] R. J. Perry, A. Harindranath, and K. G. Wilson, Phys Rev. Lett. **65**, 2959 (1990); O. Abe, K. Tanaka, and K. G. Wilson, Phys. Rev. **D48**, 4856 (1993); in *QCD Vacuum Structure*, Paris, France (World Scientific Publishing Co., Singapore, 1993), p. 229; in *Quest for Links to New Physics*, Kazimierz, Poland (World Scientific Publishing Co., Singapore, 1993), p. 508.
- [4] O. Abe and K. Tanaka, in *Group Theoretical Methods in Physics*, Toyonaka, Japan (World Scientific Publishing Co., Singapore, 1994), p. 465.
- [5] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
- [6] O. Abe and K. Tanaka, in *Proceedings of the Third International Workshop on Light Cone QCD*, INFN, Laboratori Nazionali del Gran Sasso, Assegi, Italy, 1993.
- [7] T. Sugihara, M. Matsuzaki, and M. Yahiro, Phys. Rev. **D50**, 5274 (1994).
- [8] K. Hornbostel, S. J. Brodsky, and H.-C. Pauli, Phys. Rev. **D41**, 3814 (1990). M/g in Table I should be read as $M^2/(g^2/\pi + m^2)$. We are grateful to Dr. Hornbostel for informing us of this.
- [9] C. J. Hamer, Nucl. Phys. **B195**, 503 (1982).

Figure Captions

Fig. 1: The solutions to the integral equations for M/g vs. m/g in the present paper with the wavefunction expanded to order x^2 , x^4 , and x^6 . The dotted line indicates the relation $M/g = \sqrt{6} m/g$.

Fig. 2: To order x^4 , the wavefunction is $b_0(x) = 1 + \bar{A}x^2 + \bar{B}x^4$. When r and \tilde{g} are eliminated from the matches, the parameters \bar{A} and \bar{B} are related. The real solutions are constrained as shown.

Fig. 3: Results of the present work are compared to others calculations. Filled dots indicate the results of the present work, circles show the Hornbostel *et al.*'s results [8], open squares present the results of 't Hooft's large- N expansion [5], and diamonds indicate Sugihara *et al.*'s results [7]. The masses M and m are given in units of $\sqrt{g^2 N_c / 2\pi}$.

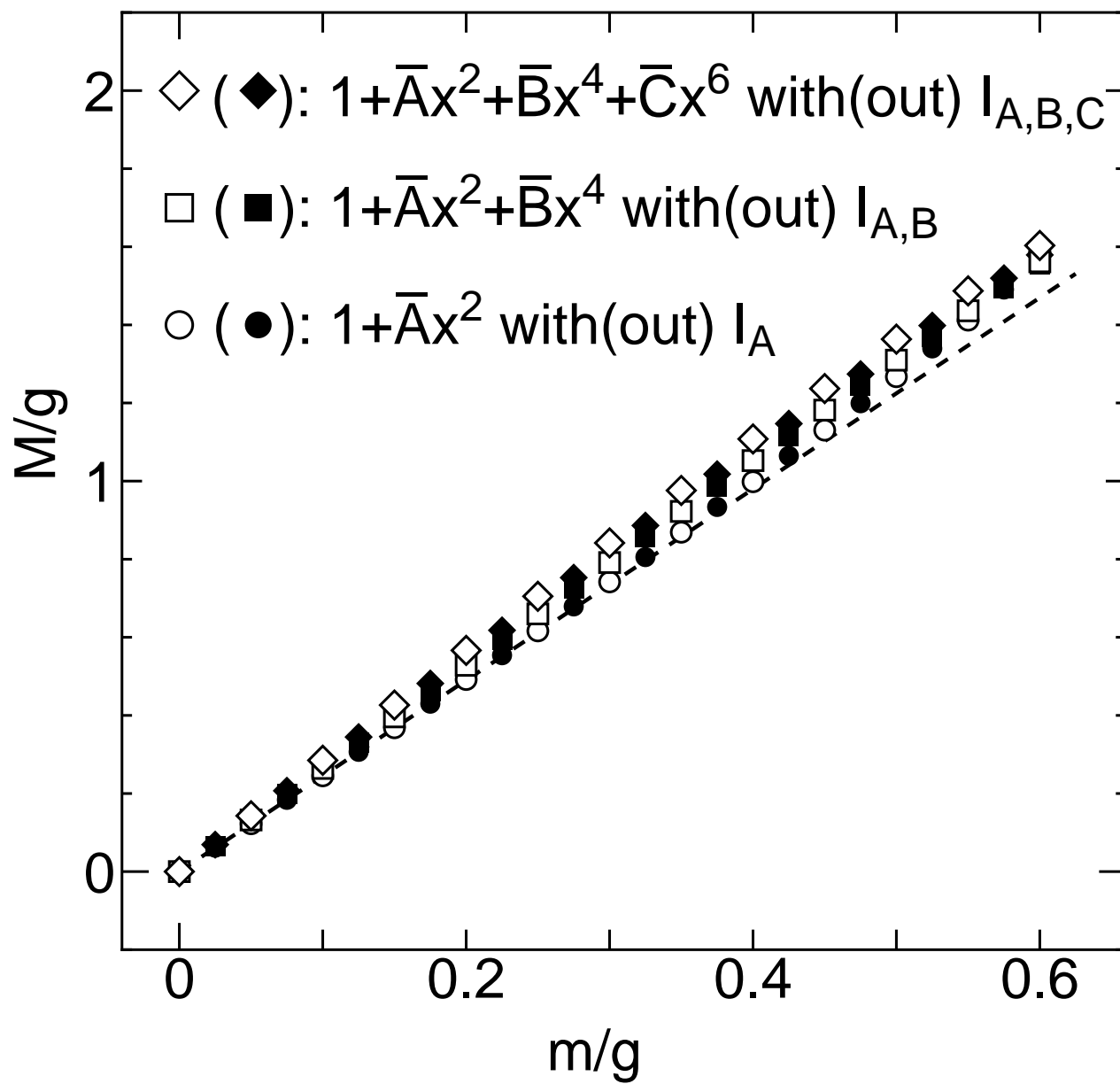


Fig. 1

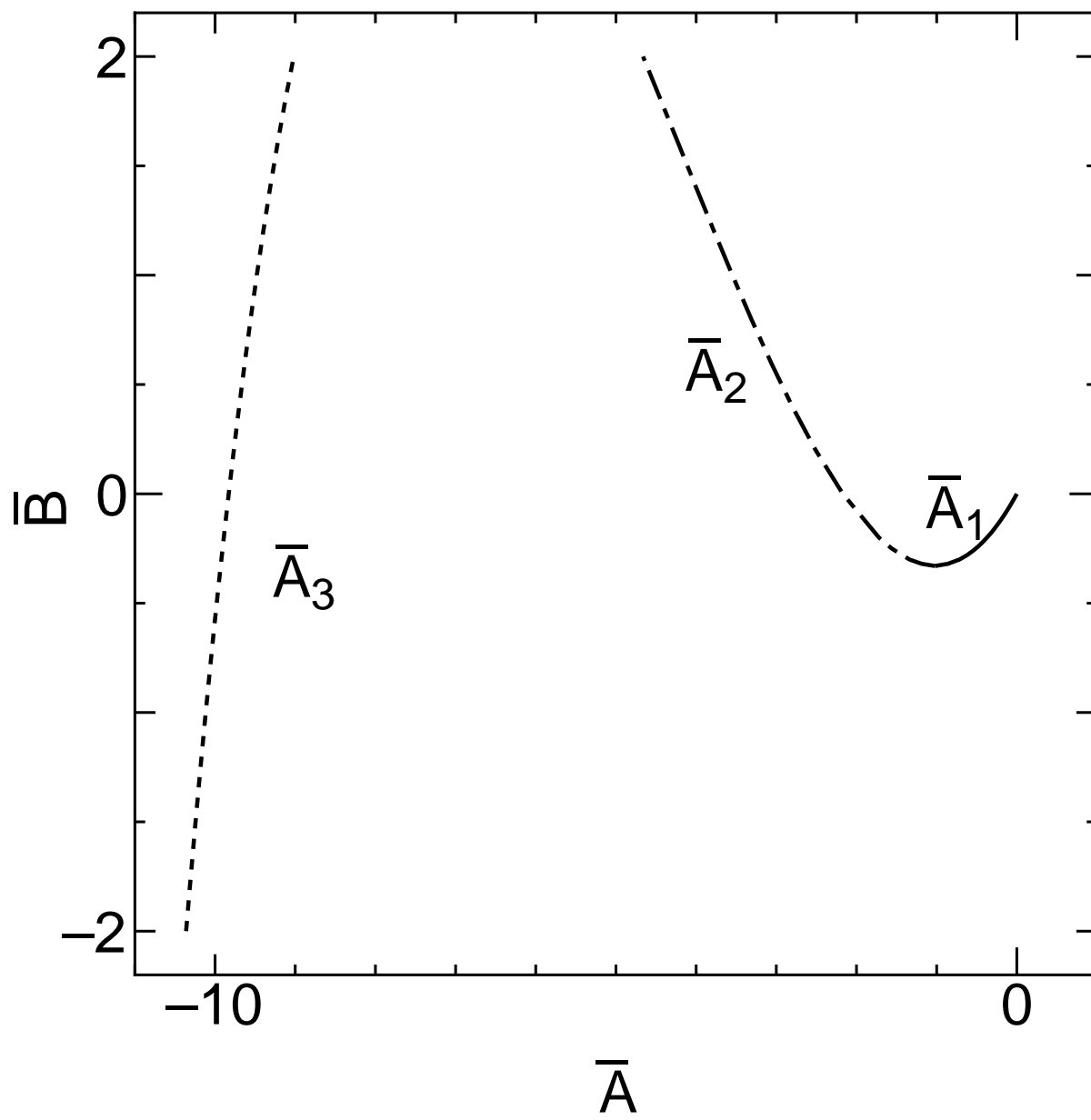


Fig. 2

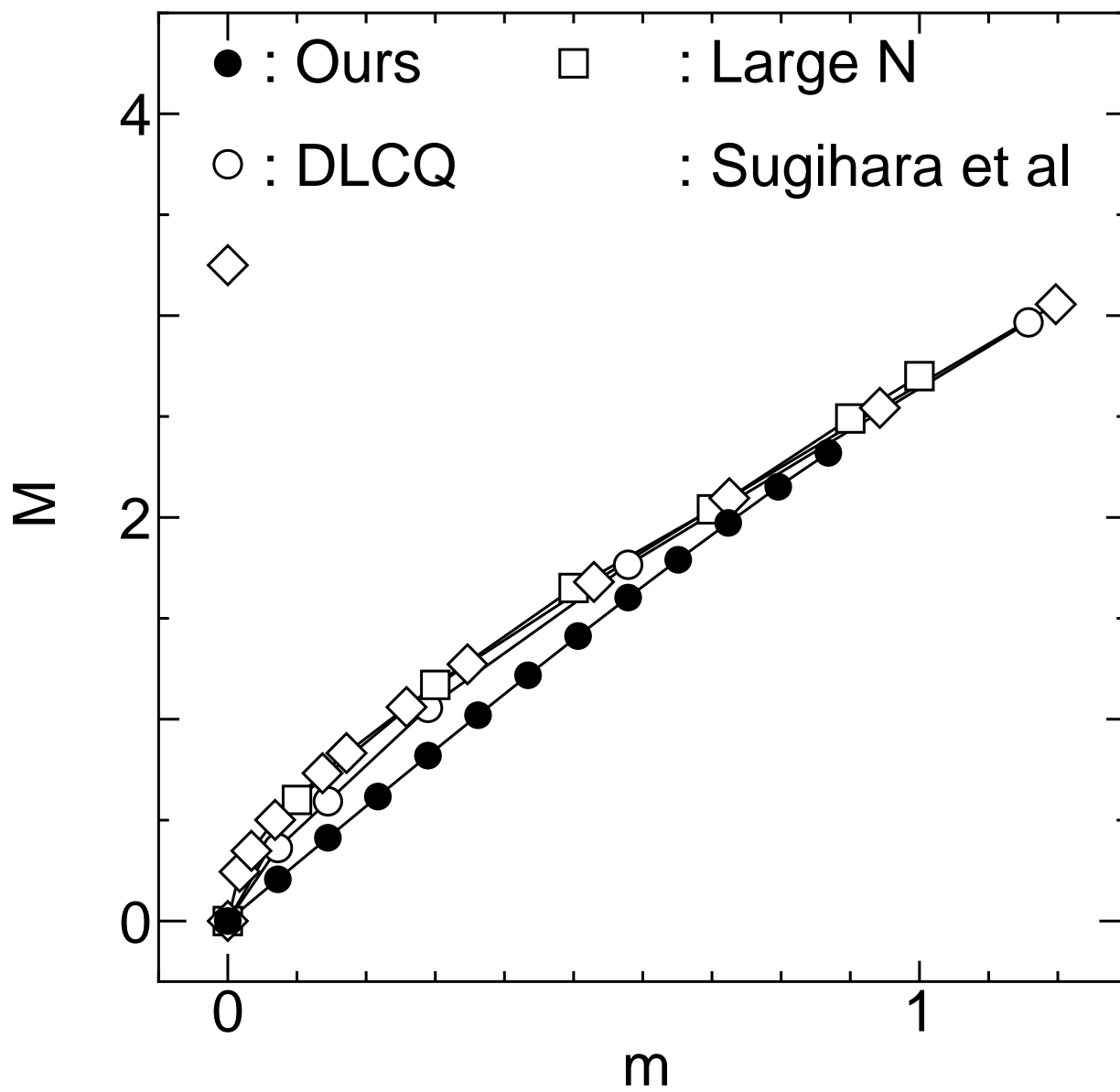


Fig. 3